Convexity Adjustment

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November 12, 2007

1 Introduction

In this short note it is described how to calculate the value today of a floating rate payment, i.e. the value of a payment which size first will be revealed at a future time. To do this in a theoretical correct manner it is necessary to introduce the concept of convexity adjustment. First, however, some notation will be introduced.

T_S	Fixing-start date
T_E	Fixing-end date
T_P	Pay-off date
$\Delta_{S,E}$	Coverage for the period $[T_S;T_E]$, i.e. length of period, in years, between time T_S and T_E using the day-count-basis convention ACT/360
$P(S,T)$	Zero-coupon price at time S for the maturity T
$F(t,T_S,T_E)$	Forward rate at time t for the rate that is fixing at time T_S over the period $[T_S;T_E]$
$V_0(F(T_S;T_S,T_E))$	Value of today of $F(T_S;T_S,T_E)$
$\mathbb{E}_0^{\mathbb{T}_E}[X]$	Expectation today under the $\mathbb{Q}^{\mathbb{T}_E}$-measure of the the stochastic variable X

Table 1: Notation

In general, the forward rate at time t that fixes over the period $[T_S;T_E]$ is defined as

$$F(t;T_S,T_E) = \frac{P(t,T_S) - P(t,T_E)}{\Delta_{S,E}P(t,T_E)} = \frac{1}{\Delta_{S,E}}\left(\frac{P(t,T_S)}{P(t,T_E)} - 1\right) \tag{1}$$

From this definition we can see that the forward rate is a martingale under the terminal measure $\mathbb{Q}^{\mathbb{T}_E}$. The reason is that the forward rate is defined as the difference between two traded securities, $P(t,T_S)$ and $P(t,T_E)$, divided by the numeraire for the $\mathbb{Q}^{\mathbb{T}_E}$-measure, $P(t,T_E)$. Thus, by definition, forward rates having fixing-end date at T_E are martingales under the $\mathbb{Q}^{\mathbb{T}_E}$-measure. This is an important feature of forward rates because it means that for any s,t with $s < t < T_S$

$$\mathbb{E}_s^{\mathbb{T}_E}\left[F(t;T_S,T_E)\right] = F(s;T_S,T_E) \tag{2}$$

Especially, by putting $s = 0$ and $t = T_S$, we have that the expected value, under the $\mathbb{Q}^{\mathbb{T}_E}$-measure, of the forward rate that fixes at time T_S over the period $[T_S;T_E]$ is given by todays forward rate for the period.

2 Risk-Neutral Pricing

To determine the value today of the payment $F(T_S;T_S,T_E)$ we resolve to risk-neutral pricing, which states that the value today of a payment X_P that is paid out at time T_P is given by

$$V_0\left(X_P\right) = P(0,T_P)\mathbb{E}_0^{\mathbb{T}_P}\left[\frac{X_P}{P(T_P,T_P)}\right] = P(0,T_P)\mathbb{E}_0^{\mathbb{T}_P}\left[X_P\right] \tag{3}$$

Using Equation (3) to determine the value today of $F(T_S;T_S,T_E)$ paid out at time T_E is now extremely easy. We have

$$\begin{aligned} V_0\left(F(T_S;T_S,T_E)\right) &= P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\right] \\ &= P(0,T_E)F(0;T_S,T_E) \end{aligned} \tag{4}$$

The first line in Equation (4) is simply an application of Equation (3) and the second line is an application of the martingale property of the forward rate under the terminal measure, Equation (2).

Taking a closer look at Equation (4) we see that the result heavily depends on $F(T_S;T_S,T_E)$ being a martingale under the relevant pricing measure, which is a result of the pay-off date being equal to the fixing-end date, $T_P = T_E$. What happens if the pay-off date is different from the fixing-end date? In Figure 1 and 2 we have shown the two cases where $T_P \neq T_E$.

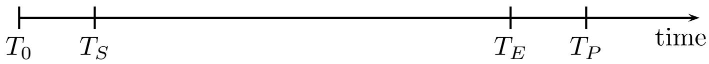

Figure 1: $T_P > T_E$

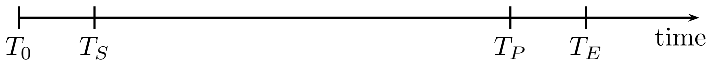

Figure 2: $T_P < T_E$

Lets assume that $T_P > T_E$. We still wants to price under the $\mathbb{Q}^{\mathbb{T}_E}$-measure because $F(T_S;T_S,T_E)$ is a martingale under that measure. But Equation (3) now becomes

$$\begin{aligned} &V_0\left(F(T_S;T_S,T_E)\right) \\ =&P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)P(T_E,T_P)\right] \\ =&P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\left(1+\Delta_{E,P}F(T_E;T_E,T_P)^{-1}\right)\right] \end{aligned} \tag{5}$$

Compared to the situation in Equation (4), we have to discount the payment from the pay-off date back to the fixing-end date under the expectation operator. This discounting of the payment complicates matter, because we now have to know the joint distribution of $F(T_S;T_S,T_E)$ and $F(T_E;T_E,T_P)$

If instead $T_P < T_E$ we must discount the payment from T_P to T_E and we get

$$\begin{aligned} &V_0\left(F(T_S;T_S,T_E)\right) \\ =&P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)P(T_P,T_E)\right] \\ =&P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\left(1+\Delta_{P,E}F(T_P;T_P,T_E)\right)\right] \end{aligned} \tag{6}$$

and now the joint distribution of $F(T_S;T_S,T_E)$ and $F(T_P;T_P,T_E)$ must be know.

In the next section we show how to adjust the forward rate so that we can calculate the expressions shown in Equation (5) and (6).

3 Convexity Adjusted Forward Rates

Case $T_P > T_E$

First we consider the case shown in Figure 1, in which $T_P > T_E$. As a starting point we take Equation (5). Now, we make the approximation $(1+x)^{-1} \approx 1 - x$ for small x to get rid of the $(\cdot)^{-1}$-operator. This yields

$$\begin{aligned} & V_0\left(F(T_S;T_S,T_E)\right) \\ \approx & P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\left(1-\Delta_{E,P}F(T_E;T_E,T_P)\right)\right] \end{aligned} \tag{7}$$

Another assumption is now made about the relation between the two forward rates. We will assume that $F(T_E;T_E,T_P) = \gamma F(T_S;T_S,T_E)$, where $\gamma = F(0;T_E,T_P)/F(0;T_S,T_E)$. Inserting this into Equation (7) gives

$$\begin{aligned} & V_0\left(F(T_S;T_S,T_E)\right) \\ \approx & P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\left(1-\Delta_{E,P}\gamma F(T_S;T_S,T_E)\right)\right] \end{aligned} \tag{8}$$

We thus have

$$\begin{aligned} & V_0\left(F(T_S;T_S,T_E)\right) \\ \approx & P(0,T_E)\left\{\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\right]-\mathbb{E}_0^{\mathbb{T}_E}\left[\Delta_{E,P}\gamma F(T_S;T_S,T_E)^2\right]\right\} \\ = & P(0,T_E)\left\{F(0;T_S,T_E)-\Delta_{E,P}\gamma\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)^2\right]\right\} \end{aligned} \tag{9}$$

The last term in Equation (9) is known as the convexity adjustment. It is how much the forward rate should be adjusted to take into account that the pay-off date is later then the fixing-end date.

This is as far as general pricing theory takes us. To proceed, we need to compute the second moment of $F(T_S;T_S,T_E)$ under the terminal measure, and this cannot be done without putting more structure on the model. If forward rates are log-normal, as in the log-normal forward LIBOR model, the second moment of the forward rate is given by

$$\mathbb{E}_t^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)^2\right] = F(t;T_S,T_E)^2\exp\left(\sigma(T_S,T_E)^2(T_S-t)\right) \tag{10}$$

Inserting Equation (10) into Equation (9) yields

$$\begin{aligned} & V_0\left(F(T_S;T_S,T_E)\right) \\ = & P(0,T_E)\left\{F(0;T_S,T_E)-\Delta_{E,P}\gamma F(0;T_S,T_E)^2\exp\left(\sigma(T_S,T_E)^2T_S\right)\right\} \end{aligned} \tag{11}$$

Case $T_P < T_E$**:**

If instead we have a situation like the one shown in Figure 2, i.e. $T_P < T_E$, the derivations are nearly the same as above. The only difference is that we do not have to make the approximation in Equation (7). Thus

$$\begin{aligned} & V_0\left(F(T_S;T_S,T_E)\right) \\ \approx & P(0,T_E)\mathbb{E}_0^{\mathbb{T}_E}\left[F(T_S;T_S,T_E)\left(1+\Delta_{P,E}F(T_P;T_P,T_E)\right)\right] \end{aligned} \tag{12}$$

The same derivations apply to this situation with the obvious change in notation. The value today of the payment $F(T_S;T_S,T_E)$ paid out at time $T_P < T_E$ is given by

$$\begin{aligned} V_0 = & P(0,T_E)\left\{F(0;T_S,T_E)\right. \\ & \left.+\Delta_{P,E}\gamma F(0;T_S,T_E)^2\exp\left(\sigma(T_S,T_E)^2T_S\right)\right\} \end{aligned} \tag{13}$$

4 Convexity Adjustment in LFM

In this section, the convexity adjustment for the log-normal forward LIBOR model (LFM) is derived - Equation (10). In the LFM, since forward rates are martingales under their terminal measure, dynamics of forward rates are assumed to follow the SDE

$$dF(t;T_S,T_E) = F(t,T_S,T_E)\sigma(T_S,T_E)dW^{\mathbb{T}_E}(t) \tag{14}$$

where $W^{\mathbb{T}_E}(t)$ is a Brownian motion under $\mathbb{Q}^{\mathbb{T}_E}$ and $\sigma(T_S,T_E)$ is the constant volatility for the forward rate covering the period $[T_S;T_E]$. To calculate the second moment of the forward we find the dynamics of the squared forward rate

$$\begin{aligned} dF(t;T_S,T_E)^2 =& 2F(t;T_S,T_E)^2\sigma(T_S,T_E)dW^{\mathbb{T}_E}(t) \\ &+ F(t;T_S,T_E)^2\sigma^2(T_S,T_E)dt \end{aligned} \tag{15}$$

The solution to Equation (15) is found by determining the dynamics in $d\ln(F(t;\cdot)^2$. This yields

$$d\left(\ln(F(t;T_s,T_E)^2)\right) = -\sigma(T_S,T_E)^2dt + 2\sigma(T_S,T_E)dW^{\mathbb{T}_E}(t) \tag{16}$$

This means that for $u > t$, $\ln(F(u;T_s,T_E)^2) \sim N(M,V^2)$, with $M = \ln(F(t;T_S,T_E)) - \sigma(T_S,T_E)^2(u-t)$ and $V^2 = 4\sigma(T_S,T_E)^2(u-t)$.

Now for any normally distributed stochastic variable, $\tilde{X}$, with mean M and variance V^2 we have

$$\mathbb{E}\left[\exp(\tilde{X})\right] = \exp\left(M + \frac{1}{2}V^2\right) \tag{17}$$

Using this result with $F(t;T_S,T_E)$ playing the role as $\tilde{X}$ yields

$$\mathbb{E}_t^{\mathbb{T}_E}\left[F(T_S,T_S,T_E)^2)\right] = F(t,T_S,T_E)^2\exp\left(\sigma(T_S,T_E)(T_S-t)\right) \tag{18}$$