# Computations in the Hull-White Model

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## 1 Specifications

In the Hull-White model, the $\mathbb{Q}$ dynamics of the spot rate is given by the following stochastic differential equation (SDE) also know as the Ohrnstein-Uhlenbeck process

$$dr(t) = (\Theta(t) - \kappa r(t))\, dt + \sigma dW(t) \tag{1}$$

where $\frac{\Theta(t)}{\kappa}$ is the long term level to which the spot rate, $r(t)$, is moving, $\kappa$ is the rate at which the spot rate rate is pushed towards the long term level and $W(t)$ is a Brownian motion under $\mathbb{Q}$. $\sigma$ is the constant volatility of changes in the spot rate. $\sigma$ is assumed constant in this note.

Using the spot rate defined by (1), we construct a money market account by

$$A(t) = e^{-\int_0^t r_u du} \tag{2}$$

$$dA(t) = -r_t A(t) dt \tag{3}$$

The specification of the spot rate, means that the Hull-White model belongs to the affine class of interest rate models and thus prices of zero coupon bonds at time $t$ for the time $T$ maturity have the following form

$$P(t,T) = e^{\alpha(t,T) + \beta(t,T) r_t} \tag{4}$$

The $T$-yield at $t$ time $t$, $y(t,T)$ is defined as

$$y(t,T) = \frac{-\ln P(t,T)}{T-t}$$

We generally want the model to be calibrated to the market today meaning that model prices of zero coupon bonds today, $P(0,T)\ \forall\ T$, is equal to the prices observed in the market. This can be achieved by choosing $\Theta(t)$ in equation (1) so that the initial yield curve is matched.

## 2 Closed Form Solution for Prices of Zero Coupon Bonds

We will now find explicit formulas for the functions $\alpha(t,T)$ and $\beta(t,T)$ in (4) and thus closed form solutions for zero coupon bonds in the Hull-White model. First, however, we derive the fundamental partial differential equation for zero coupon prices in the Hull-White model. Start by finding the dynamics of zero coupon prices by employing Ito's lemma.

$$dP(t,T) = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} dr(t) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr(t))^2$$

Inserting the spot rate dynamics (1) yields

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= \left( \frac{\partial \alpha(t,T)}{\partial t} + \frac{\partial \beta(t,T)}{\partial t} r(t) \right) dt + \beta(t,T) dr(t) + \frac{1}{2} \beta^2(t,T) (dr(t))^2 \\ &= \left( \frac{\partial \alpha(t,T)}{\partial t} + \frac{\partial \beta(t,T)}{\partial t} r(t) + \beta(t,T)\Theta(t) - \beta(t)\kappa r_t + \frac{1}{2}\beta^2(t,T)\sigma^2 \right) dt \\ &\quad + \beta(t,T)\sigma dW(t) \end{aligned}$$

We now use the dynamics of the money market account given by (3) and the dynamics of the zero coupon bonds in (5) to find the dynamics of deflated zero coupon bond prices

$$dA(t)dP(t,T) \quad = \quad A(t)dP(t,T) + P(t,T)dA(t) + \overbrace{dA(t)dP(t,T)}^{=0}$$

Again inserting what is know we get

$$\begin{aligned}\frac{dA(t)dP(t,T)}{A(t)P(t,T)} \quad = \quad & \left(\frac{\partial\alpha(t,T)}{\partial t} + \frac{\partial\beta(t,T)}{\partial t}r(t) + \beta(t,T)\Theta(t)\right. \\ & \left. -\beta(t)\kappa r_t + \frac{1}{2}\beta^2(t,T)\sigma^2 - r_t\right) dt + \beta(t,T)\sigma dW(t)\end{aligned}$$

Under the equivalent martingale measure, $\mathbb{Q}$, deflated prices are martingales. According to the martingale representation theorem we must thus have that the $dt$-term must be equal to zero, and this holds for all $t$ and $r_t$. Thus

$$\boxed{\begin{aligned}\frac{\partial\alpha(t,T)}{\partial t} + \beta(t,T)\Theta(t) + \frac{1}{2}\beta^2(t,T)\sigma^2 &= 0 &(5)\\ \alpha(T,T) &= 0 &(6)\\ \frac{\partial\beta(t,T)}{\partial t} - \kappa\beta(t,T) - 1 &= 0 &(7)\\ \beta(T,T) &= 0 &(8)\end{aligned}}$$

We solve the two ordinary differential equations by first postulating a solution for $\beta(t,T)$

$$\boxed{\beta(t,T) = \frac{1}{\kappa}\left(e^{-\kappa(T-t)} - 1\right)} \tag{9}$$

It is easy to check that the solution in (9) in fact solves the ODE in (7) subject to the boundary condition in (8). Since the derivative of $\alpha(t,T)$ only depends on $\beta(t,T)$ simple integration of $\frac{\partial\alpha(u,T)}{\partial u}$ between $t$ and $T$ is a solution to (5). Recall that

$$\int_t^T \frac{\partial\alpha(t,T)}{\partial t}du = [\alpha(u,T)]_t^T = \alpha(T,T) - \alpha(t,T) \tag{10}$$

and thus we have

$$\alpha(t,T) = \int_t^T \beta(u,T)\Theta(u)du + \frac{1}{2}\int_t^T \beta^2(u,T)\sigma^2 du \tag{11}$$

As mentioned above we wan't the model to match current zero coupon prices. This is done by choosing $\Theta(u)$ in (11) so that the initial yield curve is matched. Instead of calibrating the model to zero coupon yields directly, we calibrate the model to the term structure of forward rates. Forward rates are defined as

$$f^M(0,T) \equiv -\frac{\partial \ln P(0,T)}{\partial T} = -\frac{\partial}{\partial T}\alpha(0,T) - \frac{\partial}{\partial T}\beta(0,T)r_0$$

From (9) we get

$$\frac{\partial}{\partial T}\beta(t,T) = -e^{-\kappa(T-t)} \tag{12}$$

Using Leibniz's rule for differentiating integrals we have from (11), (6) and (8)

$$\begin{aligned}\frac{\partial}{\partial T}\alpha(0,T) &= \beta(T,T)\Theta(T)+\int_0^T \frac{\partial}{\partial T}\beta(u,T)\Theta(u)du \\ & \frac{1}{2}\beta^2(T,T)\sigma^2+\sigma^2\int_0^T \beta(u,T)\frac{\partial}{\partial T}\beta(u,T)du\end{aligned}$$

Inserting (9) and (12) yields

$$\begin{aligned}\frac{\partial}{\partial T}\alpha(0,T) &= -\int_0^T e^{-\kappa(T-u)}\Theta(u)du-\frac{\sigma^2}{\kappa}\int_0^T\left(e^{-\kappa(T-u)}-1\right)e^{-\kappa(T-u)}du \\ &= -\int_0^T e^{-\kappa(T-u)}\Theta(u)du-\frac{\sigma^2}{\kappa^2}\left[\frac{1}{2}\left(1-e^{-2\kappa T}\right)-\left(1-e^{-\kappa T}\right)\right]\end{aligned}$$

Putting things together we get

$$\begin{aligned}f^M(0,T) &= \int_0^T e^{-\kappa(T-u)}\Theta(u)du & (13)\\ &+\frac{\sigma^2}{\kappa^2}\left[\frac{1}{2}\left(1-e^{-2\kappa T}\right)-\left(1-e^{-\kappa T}\right)\right]+e^{-\kappa T}r_0 & (14)\end{aligned}$$

To isolate $\Theta(T)$ we differentiate with respect to $T$

$$\begin{aligned}\frac{\partial f^M(0,T)}{\partial T} &= \Theta(T)-\kappa\int_t^T e^{-\kappa(T-u)}\Theta(u)du \\ &+\frac{\sigma^2}{\kappa}\left(e^{-2\kappa T}-e^{-\kappa T}\right)-\kappa e^{-\kappa T}r_0\end{aligned}$$

Using (13) this can be written as

$$\begin{aligned}\frac{\partial f^M(0,T)}{\partial T} &= \Theta(T)-\kappa f^M(0,T)+\frac{\sigma^2}{\kappa}\left[\frac{1}{2}\left(1-e^{-2\kappa T)}\right)-\left(1-e^{-\kappa T)}\right)\right] \\ &+\kappa e^{-\kappa T}r_0+\frac{\sigma^2}{\kappa}\left(e^{-2\kappa T}-e^{-\kappa T}\right)-\kappa e^{-\kappa T}r_0 \\ &= \Theta(T)-\kappa f^M(0,T)-\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa T}\right)\end{aligned}$$

And thus

$$\boxed{\Theta(T)=\frac{\partial f^M(0,T)}{\partial T}+\kappa f^M(0,T)+\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa T}\right)} \qquad (15)$$

Now that we know $\Theta(T)$ we can plug it into (11) to find an expression for $\alpha(t,T)$. We compute first the integral

$$\begin{aligned}& \frac{\sigma^2}{2}\int_t^T \beta^2(u,T)du \\ =\ & \frac{\sigma^2}{2\kappa^2}\int_t^T\left(e^{-\kappa(T-u)-1}\right)^2 du \\ =\ & \frac{\sigma^2}{2\kappa^2}\left[\frac{1}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right)+(T-t)-\frac{2}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\right] & (16)\end{aligned}$$

Next we compute the integral

$$\begin{aligned}\int_t^T \beta(u,T)\Theta(u)du &= \frac{1}{\kappa}\int_t^T \left(e^{-\kappa(T-u)}-1\right)\Theta(u)du \\ &= \frac{1}{\kappa}\int_t^T e^{-\kappa(T-u)}\Theta(u)du - \frac{1}{\kappa}\int_t^T \Theta(u)du\end{aligned}$$

Inserting (15) yields

$$\begin{aligned}&\frac{1}{\kappa}\int_t^T e^{-\kappa(T-u)}\left(\frac{\partial f^M(0,u)}{\partial u}+\kappa f^M(0,u)\right)du \\ &-\frac{1}{\kappa}\int_t^T \frac{\partial f^M(0,u)}{\partial u} - \kappa f^M(0,u)du \\ &+\frac{\sigma^2}{2\kappa}\int_t^T \left(e^{-\kappa(T-u)}-1\right)\left(1-e^{-2\kappa u}\right)du \end{aligned} \tag{17}$$

Computing the last integral yields

$$\begin{aligned}&\frac{\sigma^2}{2\kappa}\int_t^T \left(e^{-\kappa(T-u)}-1\right)\left(1-e^{-2\kappa u}\right)du = \\ &\frac{\sigma^2}{2\kappa^3}\left[1-e^{-\kappa(T-t)}+\frac{1}{2}e^{-2\kappa T}-e^{-\kappa(T+t)}+\frac{1}{2}e^{-2\kappa t}\right]-\frac{\sigma^2}{2\kappa^2}(T-t)\end{aligned}$$

We now have

$$\begin{aligned}&\int_t^T \beta(u,T)\Theta(u)du = \\ &\frac{1}{\kappa}\int_t^T e^{-\kappa(T-u)}\frac{\partial f^M(0,u)}{\partial u}du + \int_t^T e^{-\kappa(T-u)}f^M(0,u)du \\ &-\frac{1}{\kappa}\left[f^M(0,u)\right]_t^T - \int_t^T f^M(0,u)du \\ &+\frac{\sigma^2}{2\kappa^3}\left[1-e^{-\kappa(T-t)}+\frac{1}{2}e^{-2\kappa T}-e^{-\kappa(T+t)}+\frac{1}{2}e^{-2\kappa t}\right] \\ &-\frac{\sigma^2}{2\kappa^2}(T-t)\end{aligned} \tag{18}$$

Now use the integration by parts formula on the first term on the right hand side in equation (18)

$$\begin{aligned}\frac{1}{\kappa}\int_t^T e^{-\kappa(T-u)}\left(\frac{\partial f^M(0,u)}{\partial u}\right)du &= \frac{1}{\kappa}\left[e^{-\kappa(T-u)}f^M(0,u)\right]_t^T \\ &\quad -\int_t^T e^{-\kappa(T-u)}f^M(0,u)du\end{aligned}$$

to get

$$\begin{aligned}&\int_t^T \beta(u,T)\Theta(u)du = \\ &\frac{1}{\kappa}\left[e^{-\kappa(T-u)}f^M(0,u)\right]_t^T - \int_t^T e^{-\kappa(T-u)}f^M(0,u)du \\ &+\int_t^T e^{-\kappa(T-u)}f^M(0,u)du - \frac{1}{\kappa}\left(f^M(0,T)-f^M(0,t)\right) - \int_t^T f^M(0,u)du \\ &+\frac{\sigma^2}{2\kappa^3}\left[1-e^{-\kappa(T-t)}+\frac{1}{2}e^{-2\kappa T}-e^{-\kappa(T+t)}+\frac{1}{2}e^{-2\kappa t}\right]-\frac{\sigma^2}{2\kappa^2}(T-t)\end{aligned} \tag{19}$$

Now simplifying gives

$$\int_t^T \beta(u,T)\Theta(u)du = -f^M(0,t)\beta(t,T) - \int_t^T f^M(0,u)du$$
$$+\frac{\sigma^2}{2\kappa^3}\left[1 - e^{-\kappa(T-t)} + \frac{1}{2}e^{-2\kappa T} - e^{-\kappa(T+t)} + \frac{1}{2}e^{-2\kappa t}\right] - \frac{\sigma^2}{2\kappa^2}(T-t) \tag{20}$$

Combining the two integrals (16) and (20) we get

$$\begin{aligned}\alpha(t,T) &= \int_t^T \beta(u,T)\Theta(u)du + \frac{\sigma^2}{2}\int_t^T \beta^2(t,T)du \\ &= -f^M(0,t)\beta(t,T) - \int_t^T f^M(0,u)du \\ &\quad +\frac{\sigma^2}{2\kappa^3}\left[1 - e^{-\kappa(T-t)} + \frac{1}{2}e^{-2\kappa T} - e^{-\kappa(T+t)} + \frac{1}{2}e^{-2\kappa t}\right] - \frac{\sigma^2}{2\kappa^2}(T-t) \\ &\quad +\frac{\sigma^2}{2\kappa^2}\left[\frac{1}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) + (T-t) - \frac{2}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right]\end{aligned}$$

Simplifying yields

$$\begin{aligned}\alpha(t,T) &= -f^M(0,t)\beta(t,T) - \int_t^T f^M(0,u)du \\ &\quad +\frac{\sigma^2}{4\kappa}\beta^2(t,T)\left(e^{-2\kappa t} - 1\right)\end{aligned} \tag{21}$$

We also have that

$$P(0,T) = e^{-\int_t^T f(0,u)du}$$

which leads to

$$\ln P(0,T) = -\int_0^T f(0,u)du$$

and thus

$$\ln\left(\frac{P(0,T)}{P(0,t)}\right) = -\int_t^T f(0,u)du$$

and we have

$$\boxed{\begin{aligned}\alpha(t,T) = &-f^M(0,t)\beta(t,T) + \ln\left(\frac{P(0,T)}{P(0,t)}\right) \\ &+ \frac{\sigma^2}{4\kappa}\beta^2(t,T)\left(e^{-2\kappa t} - 1\right)\end{aligned}} \tag{22}$$

## 3 Solving the Stochastic Differential Equation

The solution to the SDE in equation (1) can be found by employing Ito's lemma to find the dynamics of $e^{\kappa t}r(t)$ and then integrating up. This yields the solution to (1)

$$r_T = e^{-\kappa(T-t)}r_t + \int_t^T e^{-\kappa(T-u)}\Theta(u)du + \sigma\int_t^T e^{-\kappa(T-u)}dW_u$$

Since $\mathbb{E}^{\mathbb{Q}}\left[\sigma \int_t^T e^{-\kappa(T-u)} dW_u | \mathcal{F}_t\right] = 0$, we have

$$\mathbb{E}_t^{\mathbb{Q}}[r_T] = e^{-\kappa(T-t)} r_t + \int_t^T e^{-\kappa(T-u)} \Theta(u) du$$

$$\begin{aligned}\mathrm{Var}_t^{\mathbb{Q}}[r_T] &= \mathbb{E}_t^{\mathbb{Q}}\left[\left(r_t - \mathbb{E}_t^{\mathbb{Q}}[r_T]\right)^2\right] \\ &= \sigma^2 \int_t^T e^{-2\kappa(T-u)} du\end{aligned}$$

Notice that

$$\int_t^T e^{-\kappa(T-u)} \Theta(u) du = -\int_t^T \frac{\partial}{\partial T} \beta(u,T) \Theta(u) du$$

Liebniz's rule for differentiating integral gives

$$\frac{\partial}{\partial T} \int_t^T \beta(u,T)\Theta(u) du = \beta(T,T)\Theta(T) + \int_t^T \frac{\partial}{\partial T}\beta(u,T)\Theta(u) du$$

where the first term on the right hand side is equal to zero according to (8). Thus we have

$$\int_t^T e^{-\kappa(T-u)} \Theta(u) du = -\frac{\partial}{\partial T} \int_t^T \beta(u,T)\Theta(u) du$$

Differentiating (20) with respect to $T$ we get

$$\begin{aligned}\int_t^T e^{-\kappa(T-u)} \Theta(u) du &= f^M(0,t) \frac{\partial}{\partial T}\beta(t,T) + \frac{\partial}{\partial T} \int_t^T f^M(0,u) du \\ &\quad - \frac{\sigma^2}{2\kappa^2}\left(e^{-\kappa(T-t)} - e^{-2\kappa T} + e^{-\kappa(T+t)} - 1\right) \\ &= -f^M(0,t) e^{-\kappa(T-t)} + f^M(0,T) - \underbrace{\int_t^T \frac{\partial}{\partial T} f^M(0,u)}_{=0} \\ &\quad - \frac{\sigma^2}{2\kappa^2}\left(e^{-\kappa(T-t)} - e^{-2\kappa T} + e^{-\kappa(T+t)} - 1\right)\end{aligned}$$

Now add and subtract $\frac{\sigma^2}{2\kappa^2}(1-e^{-kT})^2$ and simplify to get

$$\int_t^T e^{-\kappa(T-u)} \Theta(u) du = \gamma(T) - \gamma(t) e^{-\kappa(T-t)}$$

where

$$\gamma(t) = f^M(0,t) + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-kt}\right)^2$$

Thus the conditional expectation of the future spot rate can now be written as

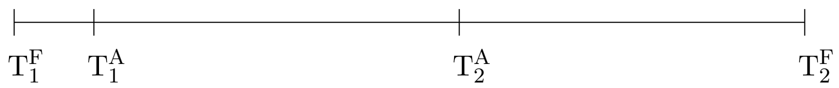
$$\boxed{\mathbb{E}_t^{\mathbb{Q}}[r_T] = e^{-\kappa(T-t)} r_t + \gamma(T) - \gamma(t) e^{-\kappa(T-t)}} \tag{23}$$

And the conditional varians is given by

$$\boxed{\mathrm{Var}_t^{\mathbb{Q}} = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)} \tag{24}$$

# 4 Floaters and Caplets

In this section the Hull-White model is used to value a future stochastic rate today. We want to compute the price of a payment received at time $T_2^A$. This payment covers interest over the period from time $T_1^A$ to $T_2^A$. The payment is not known until time $T_1^F$ where it is fixed as the simple rate over the period $T_1^F$ to $T_2^F$.



The following notation is used

$T_1^F$ : Start date of fixing period
$T_2^F$ : End date of fixing period
$T_1^A$ : Start date of accrual period
$T_2^A$ : End date of accrual period
$\Delta_1$ : Accrual period in years $(T_2^A - T_1^A)$
$\Delta_2$ : Fixing period in years $(T_2^F - T_1^F)$

At time $T_1^F$ the spot LIBOR rate over the fixing period is given by

$$L(T_1^F, T_2^F) = \frac{1}{\Delta_2}\left(\frac{1}{P(T_1^F, T_2^F)} - 1\right) \tag{25}$$

Since the accrual period is $\Delta_1$ the payment received at time $T_2^A$ is equal to $\Delta_1 L(T_1^F, T_2^F)$. The value at time $t$ of the unknown payment is given by

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_2^A} r_u\,du}\Delta_1 L(T_1^F, T_2^F)\right] \tag{26}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_1^F} r_u\,du}\Delta_1 L(T_1^F, T_2^F)\,\mathbb{E}_{T_1^F}^{\mathbb{Q}}\left[e^{-\int_{T_1^F}^{T_2^A} r_u\,du}\right]\right] \tag{27}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_1^F} r_u\,du}P(T_1^F, T_2^A)\Delta_1 L(T_1^F, T_2^F)\right] \tag{28}$$

Changing from the risk-neutral measure to the forward measure with the zero-coupon bond maturing at time $T_1^F$ as numeraire yields

$$V(t) = P(t, T_1^F)\mathbb{E}_t^{T_1^F}\left[P(T_1^F, T_2^A)\Delta_1 L(T_1^F, T_2^F)\right] \tag{29}$$

which is the same as

$$V(t) = P(t, T_1^F)\frac{\Delta_1}{\Delta_2}\mathbb{E}_t^{T_1^F}\left[\left(\frac{P(T_1^F, T_2^A)}{P(T_1^F, T_2^F)} - P(T_1^F, T_2^A)\right)\right]$$

$$= \frac{\Delta_1}{\Delta_2}\left\{P(t, T_1^F)\mathbb{E}_t^{T_1^F}\left[\left(\frac{P(T_1^F, T_2^A)}{P(T_1^F, T_2^F)}\right)\right] - P(t, T_2^A)\right\} \tag{30}$$

In the Hull-White model, the ration $P(T_1^F, T_2^A)/P(T_1^F, T_2^F)$ is equal to

$$\frac{P(T_1^F, T_2^A)}{P(T_1^F, T_2^F)} = e^{\alpha(T_1^F, T_2^A) - \alpha(T_1^F, T_2^F) + (\beta(T_1^F, T_2^A) - \beta(T_1^F, T_2^F))r_{T_1^F}}$$

Which is used to get

$$V(t) = \frac{\Delta_1}{\Delta_2}\left\{P(t,\mathrm{T}_1^{\mathrm{F}})e^{\alpha(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}})-\alpha(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})}\mathbb{E}_t^{\mathrm{T}_1^{\mathrm{F}}}\left[e^{(\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}})-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}))r_{\mathrm{T}_1^{\mathrm{F}}}}\right] - P(t,\mathrm{T}_2^{\mathrm{A}})\right\} \tag{31}$$

The only unknown object in Equation (31) is $\mathbb{E}_t^{\mathrm{T}_1^{\mathrm{F}}}\left[e^{(\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}})-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}))r_{\mathrm{T}_1^{\mathrm{F}}}}\right]$, but this expectation can easily be computed since $r_{\mathrm{T}_1^{\mathrm{F}}}$ is normally distributed.

$$\begin{aligned}
&\mathbb{E}_t^{\mathrm{T}_1^{\mathrm{F}}}\left[\exp\left((\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}}) - \beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}))r_{\mathrm{T}_1^{\mathrm{F}}}\right)\right] = \\
&\exp\left((\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}}) - \beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}))\mathbb{E}_t^{\mathrm{T}_1^{\mathrm{F}}}[r_{\mathrm{T}_1^{\mathrm{F}}}] + \frac{1}{2}(\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}}) - \beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}))^2\mathrm{Var}_t^{\mathrm{T}_1^{\mathrm{F}}}[r_{\mathrm{T}_1^{\mathrm{F}}}]\right)
\end{aligned} \tag{32}$$

which can be inserted into (31) to get a closed form solution for the value of a single float payment.

**Fixing at an Average Rate**

$$\begin{aligned}
V(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{\mathrm{T}_2^{\mathrm{A}}} r_u\,du}\frac{\Delta_1}{M}\sum_{i=1}^{M} L(\mathrm{T}_{1i}^{\mathrm{F}},\mathrm{T}_{2i}^{\mathrm{F}})\right] \\
&= \frac{\Delta_1}{M}\sum_{i=1}^{M}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{\mathrm{T}_2^{\mathrm{A}}} r_u\,du}L(\mathrm{T}_{1i}^{\mathrm{F}},\mathrm{T}_{2i}^{\mathrm{F}})\right]
\end{aligned}$$

The last expression has exactly the same form as the expression found in the previous section and it's value is thus know in closed form.

**Caplet Pricing with Unnatural Time Lag** We will now find the value of a caplet that caps the simple compounded interest rate given in Equation (25). The rate is fixed over the period from $\mathrm{T}_1^{\mathrm{F}}$ to $\mathrm{T}_2^{\mathrm{F}}$ but it is paid out at time $\mathrm{T}_2^{\mathrm{A}}$.

$$\begin{aligned}
\mathrm{cpl}(t,\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}},\mathrm{T}_2^{\mathrm{A}}) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{\mathrm{T}_2^{\mathrm{A}}} r_u\,du}\Delta_1\left(L(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}) - K\right)^+\right] \\
&= \Delta_1 P(t,\mathrm{T}_2^{\mathrm{A}})\mathbb{E}_t^{\mathrm{T}_2^{\mathrm{A}}}\left[\left(L(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}}) - K\right)^+\right] \\
&= \frac{\Delta_1}{\Delta_2}P(t,\mathrm{T}_2^{\mathrm{A}})\mathbb{E}_t^{\mathrm{T}_2^{\mathrm{A}}}\left[\left(\frac{1}{P(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})} - (1+\Delta_2 K)\right)^+\right] \\
&= \frac{\Delta_1}{\Delta_2}P(t,\mathrm{T}_2^{\mathrm{A}})\mathbb{E}_t^{\mathrm{T}_2^{\mathrm{A}}}\left[\left(e^{-\alpha(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})r_{\mathrm{T}_1^{\mathrm{F}}}} - (1+\Delta_2 K)\right)^+\right] \\
&= \frac{\Delta_1}{\Delta_2}P(t,\mathrm{T}_2^{\mathrm{A}})e^{-\alpha(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})}\mathbb{E}_t^{\mathrm{T}_2^{\mathrm{A}}}\left[\left(e^{-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})r_{\mathrm{T}_1^{\mathrm{F}}}} - e^{\alpha(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})}(1+\Delta_2 K)\right)^+\right]
\end{aligned}$$

Since $r_{\mathrm{T}_1^{\mathrm{F}}} \sim \Phi(M_r, V_r^2)$, where $M_r$ and $V_r^2$ is the mean and variance of $r_{\mathrm{T}_1^{\mathrm{F}}}$ respectively and $\Phi$ is the standard cumulative normal distribution function, we have $-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})r_{\mathrm{T}_1^{\mathrm{F}}} \sim \Phi(-\beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})M_r, \beta(\mathrm{T}_1^{\mathrm{F}},\mathrm{T}_2^{\mathrm{F}})^2V_r^2)$. From Brigo and Mercurio (2001) we have the following for a lognormally distributed stochastic variable $X$ with mean $M$ and variance $V$

$$\mathbb{E}\left[(X-K)^+\right] = e^{M+\frac{1}{2}V^2}\Phi\left(\frac{M-\ln(K)+V^2}{V}\right) - K\Phi\left(\frac{M-\ln(K)}{V}\right) \tag{33}$$

The caplet value can now be computed with $M = -\beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})M_r$ and $V = \beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})V_r$ which yields

$$\begin{aligned}\mathrm{cpl}(t, \mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{A}}) = \frac{\Delta_1}{\Delta_2} P(t, \mathrm{T}_2^{\mathrm{A}}) e^{-\alpha(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})} \Bigg( & e^{-\beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})M_r + \frac{1}{2}\beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})^2 V_r^2} \Phi\left(d_1\right) \\ & - \left(e^{\alpha(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})}(1 + \Delta_2 K)\right) \Phi\left(d_1 - \beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})V_r\right) \Bigg) \end{aligned} \tag{34}$$

where

$$d_1 = \frac{-\beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})M_r - \ln(e^{\alpha(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})}(1 + \Delta_2 K)) + \beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})^2 V_r^2}{\beta(\mathrm{T}_1^{\mathrm{F}}, \mathrm{T}_2^{\mathrm{F}})V_r} \tag{35}$$

## References

Brigo, D. and F. Mercurio (2001): *Interest Rate Models Theory and Practice,* Springer, first edition.