Day-To-Day Return Decomposition

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1 Introduction

This short note describes how to decompose a day-to-day bond return into smaller components. We divide the return into a interest rate part, a redemption part and a clean price part, which is further divided into a theta component, a curve component, a volatility component, a yield-curve-spread component and finally a residual component.

2 Computation of Day-To-Day Return

A day-to-day bond return is computed on a day-to-day basis where it is assumed that the position is valued and registered each day. Any cash flows stemming from the position is re-invested in the bond.

We introduce the follwing notation:

N_t	: Accumulated number of bonds at time t - $N_0 = 1$
N_t^U	: Accumulated number of bonds at time t resulting from redemptions - $N_0^U = 0$
N_t^I	: Accumulated number of bonds at time t resulting from interests - $N_0^I = 0$
U_t	: Redemptions (of a notional of 1) received at time t
c_t	: Coupon of the bond for the relevant accrual
frq	: Payment frequency of the bond
k_t	: Dirty price at time t
v_t	: Accrued interest at time t
W_t	: Total wealth at time t - $W_0 = k_0 + v_0$
W_t^U	: Total wealth at time t stemming from redemption payments - $W_0^U = 0$
W_t^I	: Total wealth at time t stemming from interest payments - $W_0^I = 0$
R_{tT}	: Total return over the period t to T
R_{tT}^U	: Return over the period t to T stemming from redemptions
R_{tT}^I	: Return over the period t to T stemming from interests
R_{tT}^P	: Return over the period t to T stemming from changes in clean price
$D(t,T)$	: Discount factor for the period t to T

Note that N_t^I can only be positive but N_t^U can both be positive as well as negative since redemptions are made at par regardless of the price of the bond.

To compute the wealth W_t at time t we must take both redemptions payments and interest payments into account. When a redemption payment of U_t is announced to be paid at time T we can make the following decomposition of the value the bond $(k_t + v_t) = (1 - U_t)(k_t + v_t) + U_t(k_t + v_t)$. The value at time t of the last term is equal to $100U_t(1 + \frac{c_t}{frq})D(t,T)$, and thus the value of the position after a redemption announcement is equal to $(1 - U_t)(k_t + v_t) + 100U_t(1 + \frac{c_t}{frq})D(t,T)$. When a interest payment is due the value of the position is $(k_t + v_t) + 100\frac{c_t}{frq}$. In total we have

$$W_t = W_{t-1}\left(\frac{(1-U_t)(k_t+v_t) + 100U_tD(t,T)\left[1+\frac{c_t}{frq}\right] + 100\frac{c_t}{frq}}{k_{t-1}+v_{t-1}}\right) \tag{1}$$

where, with at little abuse of notation, $U_t = 0$ unless at dates where redemption payments are announced and $c_t = 0$ except for interest payment dates. This notation is used in the rest of the note.

The total return over any period $[t;T]$ is now found by

$$R_{tT} = \frac{W_T}{W_t} - 1 \tag{2}$$

3 Return Decomposition

To decompose the return in (2) we must keep track on how much redemption payments, interest payments and the clean price contributes to the return. To do this we must each day update the three accounts N_t^U, N_t^I and N_t.

The following relations hold by definition:

$$N_t^U = N_{t-1}^U + \frac{100U_tN_{t-1}D(t,T)}{k_t+v_t} - U_tN_{t-1} \tag{3}$$

At time t, we know that at time T we will receive a payment of $100U_tN_{t-1}$ due to redemptions. To find the time t value of this payment we must multiply by the discount factor for the period $D(t,T)$. We reinvest this payment in the bond at price $(k_t + v_t)$ and thus the number of bond increases by $\frac{100U_tN_{t-1}D(t,T)}{k_t+v_t}$. To get the accumulated number of bonds stemming from redemptions we add N_{t-1}^U and subtract U_tN_{t-1}.

We also have

$$N_t^I = N_{t-1}^I + \frac{100\frac{c_t}{frq}N_{t-1}\left[1+U_tD(t,T)\right]}{k_t+v_t} \tag{4}$$

Again, we reinvest all proceeds in the bond and we can thus buy $\frac{100\frac{c_t}{frq}N_{t-1}[1+U_tD(t,T)]}{k_t+v_t}$ number of bonds. The total account of bonds due to interests is found by adding the account total for the previous period.

The total number of bonds is given by:

$$N_t = N_{t-1} + \Delta N_t^U + \Delta N_t^I \tag{5}$$

$$= N_{t-1}\left[1 - U_t + \frac{100U_tD(t,T)}{k_t+v_t} + \frac{100\frac{c_t}{frq}N_{t-1}\left[1+U_tD(t,T)\right]}{k_t+v_t}\right] \tag{6}$$

where ΔN_t^U and ΔN_t^I are found by subtracting N_{t-1}^U and N_{t-1}^I from (3) and (4) respectively.

The value of a position can now easily be calculated by multiplying the size of the position with the current market price. I.e. the value at time t is given by $W_t = N_t(k_t + v_t)$ and the return for the period $[t;T]$ is given by (2). This return consists of three components - a redemption part, an interest part and a clean price part. The redemption part is given by:

$$R_{tT}^U = \frac{W_T^U}{W_t} = \frac{N_T^U(k_T + v_T)}{k_t + v_t} \tag{7}$$

and the interest part of the return is given by:

$$R_{tT}^I = \frac{W_T^I}{W_t} = \frac{N_T^I(k_T + v_T)}{k_t + v_t} \tag{8}$$

The clean price part of the return is found as a residual and is given by:

$$\begin{aligned} R_{tT}^P &= R_{tT} - R_{tT}^U - R_{tT}^I \\ &= \frac{N_T(k_T + v_T) - (k_t + v_t) - N_T^U(k_T + v_T) - N_T^I(k_T + v_T)}{k_t + v_t} \\ &= \frac{(N_T - N_T^U - N_T^I)k_T - k_t + (N_T - N_T^U - N_T^I)v_T - v_t}{k_t + v_t} \end{aligned} \tag{9}$$

Since (9) includes accrued interests, we subtract it from Equation (9) and add it to Equation (8). Thus we end up with the following relations:

$$R_{tT} = \frac{N_T(k_T + v_T)}{k_t + v_t} - 1 \tag{10}$$

$$R_{tT}^U = \frac{N_T^U(k_T + v_T)}{k_t + v_t} \tag{11}$$

$$R_{tT}^I = \frac{N_T^I(k_T + v_T) + (N_T - N_T^U - N_T^I)v_T - v_t}{k_t + v_t} \tag{12}$$

$$R_{tT}^P = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{k_t + v_t} \tag{13}$$

The total return has now been divided into a redemption return, an interest return and a clean price return since we have

$$R_{tT} = R_{tT}^U + R_{tT}^I + R_{tT}^P \tag{14}$$

4 Price Decomposition

The return stemming from changes in the clean price, R_{tT}^P, can further be divided into components stemming from passing of time (theta), curve changes (crv), volatility changes (vol), changes in the yield curve spread (ycs) and a residual. This section shows how to compute the different components that makes up the clean price return.

For at bond we will define the clean price function, $PV(\cdot)$, as the function that returns the clean price at a given date taking as input relevant market data. We thus have

$$k_t = PV(t, crv_t, vol_t, ycs_t) \tag{15}$$

In (15) t is the calendar date at which the pricing is done, crv_t is the yield curve at time t, vol_t is the volatility at time t to which the stochastic interest rate model is calibrated. Sometimes a stochastic interest rate model is unnecessary which for example is the case when future payments are deterministic, but for completeness we take it as input. ycs_t is the spread to the yield curve at time t that makes the theoretic price equal to the market price. In the case of callable bonds OAS[1] is the correct input but in general the input is the ycs.

The objective is now to isolate the effects on the return for each of the pricing inputs. To do this we make the following definitions:

Theta contribution

$$\Delta_t^{\text{theta}} = k_t - PV(t-1, crv_t, vol_t, ycs_t) \tag{16}$$

Curve contribution

$$\Delta_t^{\text{crv}} = k_t - PV(t, crv_{t-1}, vol_t, ycs_t) \tag{17}$$

Volatility contribution

$$\Delta_t^{\text{vol}} = k_t - PV(t, crv_t, vol_{t-1}, ycs_t) \tag{18}$$

YCS contribution

$$\Delta_t^{\text{ycs}} = k_t - PV(t, crv_t, vol_t, ycs_{t-1}) \tag{19}$$

Residual

$$\Delta_t^{\text{residual}} = k_t - k_{t-1} - \Delta_t^{\text{theta}} - \Delta_t^{\text{crv}} - \Delta_t^{\text{vol}} - \Delta_t^{\text{ycs}} \tag{20}$$

By adding up the day-to-day changes of the above contributions we get the difference in clean price over the period $[t; T]$, i.e. $k_T - k_t = \sum_{j=t}^{T} \Delta_j^{\text{theta}} + \Delta_j^{\text{crv}} + \Delta_j^{\text{vol}} + \Delta_j^{\text{ycs}} + \Delta_j^{\text{residual}}$

The returns stemming from these changes are defined as follows:

[1] Option Adjusted Spread

$$R_{tT}^{\text{theta}} = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)} \frac{\sum_{j=t}^T \Delta_j^{\text{theta}}}{k_t + v_t} \tag{21}$$

$$R_{tT}^{\text{crv}} = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)} \frac{\sum_{j=t}^T \Delta_j^{\text{crv}}}{k_t + v_t} \tag{22}$$

$$R_{tT}^{\text{vol}} = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)} \frac{\sum_{j=t}^T \Delta_j^{\text{vol}}}{k_t + v_t} \tag{23}$$

$$R_{tT}^{\text{ycs}} = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)} \frac{\sum_{j=t}^T \Delta_j^{\text{ycs}}}{k_t + v_t} \tag{24}$$

$$R_{tT}^{\text{residual}} = \frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)} \frac{\sum_{j=t}^T \Delta_j^{\text{residual}}}{k_t + v_t} \tag{25}$$

$$\tag{26}$$

These returns are normalized by the factor $\frac{(N_T - N_T^U - N_T^I)k_T - k_t}{(k_T - k_t)}$ since we want to explain the price difference $(N_T - N_T^U - N_T^I)k_T - k_t$ and not $k_T - k_t$. The result is that we have $R_{tT}^P = R_{tT}^{\text{theta}} + R_{tT}^{\text{crv}} + R_{tT}^{\text{vol}} + R_{tT}^{\text{ycs}} + R_{tT}^{\text{residual}}$. Combining this with Equation (14) we thus have the decomposition

$$R_{tT} = R_{tT}^U + R_{tT}^I + R_{tT}^{\text{theta}} + R_{tT}^{\text{crv}} + R_{tT}^{\text{vol}} + R_{tT}^{\text{ycs}} + R_{tT}^{\text{residual}} \tag{27}$$